CERTAIN CONTACT PROBLEMS FOR AN ELASTIC LAYER

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References [1,2] examined the indentation formed by the axial force of a rigid stamp acting on an elastic layer lying on a rigid base, under the assumption of the absence of friction between the stamp and the layer, and likewise between the layer and the base. In [3] the contact problem for an elastic layer was solved in the general case when the line of action of the force does not coincide with the axis of the stamp.*

We give below the solution to the general mixed problem for an elastic layer in the case when on one of the bounding planes there exists a circular line of separation in the boundary conditions (Section 1). The results that are obtained are applied to contact problems for an elastic layer in two cases: a) friction between the layer and the base is absent (Section 2), and b) the layer is rigidly attached to the base (Section 3); in both cases friction between the stamp and the layer is neglected.

The relations between the displacements of the stamp and the applied forces are found, and corresponding numerical data for various values of the ratio of the stamp radius to the layer thickness are presented.

1. The mixed problem for an elastic layer in the presence of a circular line of separation in the boundary conditions on one of the faces. We examine the elastic equilibrium of an unbounded layer $0 < r < \infty$, $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant z \leqslant h$ (r, ϕ, z are cylindrical coordinates) under the following conditions: the normal displacements w

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* After this paper was submitted for publication, [7] was published, in which the problem of the action of a stamp on a layer (without friction) was solved by a different method which yields a solution in the form of a power series in $1/h$, where h is the thickness of the layer.

are given on the face $z = h$; in the interior of some circle of radius $r = a$ on the face $z = 0$ the displacements w are also given, while in the exterior the normal stresses σ_z are given; finally, on both bounding surfaces of the layer the shear stresses τ_{xz} and τ_{yz} are prescribed. Hence the problem consists of obtaining the solution of the equations of elasticity under the following boundary conditions:

$$w = w_h(r, \varphi), \qquad \tau_{xz} = \tau_{xh}(r, \varphi), \qquad \tau_{yz} = \tau_{yh}(r, \varphi) \quad \text{for } z = h \tag{1.1}$$

$$w = w_0(r, \varphi) \quad \text{for } z = 0,\ r < a, \qquad \sigma_z = \sigma_0(r, \varphi) \quad \text{for } z = 0,\ r > a \tag{1.2}$$

$$\tau_{xz} = \tau_{x0}(r, \varphi), \qquad \tau_{yz} = \tau_{y0}(r, \varphi) \quad \text{for } z = 0 \tag{1.3}$$

In the solution we use the Papkovich-Neuber formulas, which give the solution of the Lame equations in the form of four harmonic functions Φ_0, Φ_1, Φ_2, Φ_3, namely

$$2\mu u = -\frac{\partial F}{\partial x} + 4(1-\nu)\Phi_1, \quad 2\mu v = -\frac{\partial F}{\partial y} + 4(1-\nu)\Phi_2,$$
$$2\mu w = -\frac{\partial F}{\partial z} + 4(1-\nu)\Phi_3 \tag{1.4}$$

Here μ is the shear modulus, and $F = \Phi_0 + x\Phi_1 + y\Phi_2 + z\Phi_3$.

We likewise introduce the formulas for the stresses entering into the boundary conditions:

$$\sigma_z = \frac{\partial}{\partial z}[2(1-\nu)\Phi_3 - \Phi_4] + 2\nu\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right) - \left(x\frac{\partial^2\Phi_1}{\partial x^2} + y\frac{\partial^2\Phi_2}{\partial y^2} + z\frac{\partial^2\Phi_3}{\partial z^2}\right)$$
$$\tau_{zx} = \frac{\partial \Phi}{\partial x} + 2(1-\nu)\frac{\partial \Phi_1}{\partial z}, \qquad \tau_{yz} = \frac{\partial \Phi}{\partial y} + 2(1-\nu)\frac{\partial \Phi_2}{\partial z} \tag{1.5}$$
$$\Phi = (1-2\nu)\Phi_3 - \Phi_4 - x\frac{\partial \Phi_1}{\partial z} - y\frac{\partial \Phi_2}{\partial z} - z\frac{\partial \Phi_3}{\partial z}, \qquad \Phi_4 = \frac{\partial \Phi_0}{\partial z}$$

where ν is Poisson's ratio.

Making use of the presence of an "extra" function in the Papkovich-Neuber solution, we supplement relationships to

$$\Phi = 0 \quad \text{for } z = 0, \qquad \Phi = 0 \quad \text{for } z = h \tag{1.6}$$

the conditions (1.1) to (1.3).

Then from the boundary conditions associated with the shear stresses we obtain two separate Neumann problems for the functions Φ_1 and Φ_2, as a consequence of which we consider these functions to be known in the sequel.

The remaining boundary conditions of the problem can be satisfied if the harmonic functions Φ_3 and Φ_4 are subjected to the conditions (in this it is assumed that all of the unknown functions are of the order r^{-1} at infinity)

$$[\Phi_3]_{z=h} = \frac{\mu}{1-\nu} w_h, \qquad \left[\Phi_4 + h\frac{\partial\Phi_3}{\partial z}\right]_{z=h} = \frac{1-2\nu}{1-\nu}\mu w_h + \frac{x\tau_{xh} + y\tau_{yh}}{2(1-\nu)} \tag{1.7}$$

$$[(1-2\nu)\Phi_3 - \Phi_4]_{z=0} = \frac{x\tau_{x0} + y\tau_{y0}}{2(1-\nu)} \tag{1.8}$$

$$[(3-4\nu)\Phi_3 - \Phi_4]_{\substack{z=0\\ r<a}} = 2\mu w_0 + \frac{x\tau_{x0} + y\tau_{y0}}{2(1-\nu)} \tag{1.9}$$

$$\left[2(1-\nu)\frac{\partial\Phi_3}{\partial z} - \frac{\partial\Phi_4}{\partial z}\right]_{\substack{z=0\\ r>a}} = \sigma_0 - 2\nu\left[\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y}\right]_{z=0} + \left[x\frac{\partial^2\Phi_1}{\partial z^2} + y\frac{\partial^2\Phi_2}{\partial z^2}\right]_{z=0} \tag{1.10}$$

We represent the harmonic functions Φ_3 and Φ_4 in the form

$$\Phi_3 = \sum_{n=-\infty}^{\infty} e^{in\varphi}\int_0^\infty [A_n \sinh\lambda(h-z) + C_n\cosh\lambda(h-z)]\, J_n(\lambda r)\frac{d\lambda}{\sinh\lambda h}$$

$$\Phi_4 = \sum_{n=-\infty}^{\infty} e^{in\varphi}\int_0^\infty [(\lambda h A_n + D_n)\cosh\lambda(h-z) + B_n\sinh\lambda(h-z)]\, J_n(\lambda r)\frac{d\lambda}{\sinh\lambda h} \tag{1.11}$$

By means of the Fourier and Hankel transforms*, one finds from condition (1.7)

$$C_n(\lambda) = \frac{\mu}{1-\nu}\lambda\sinh\lambda h\int_0^\infty w_h^{(n)}(r)\, J_n(\lambda r)\, r dr \tag{1.12}$$

$$D_n(\lambda) = \frac{\lambda\sinh\lambda h}{1-\nu}\int_0^\infty \left[(1-2\nu)\mu w_h^{(n)}(r) + \frac{1}{2}(x\tau_{xh} + y\tau_{yh})^{(n)}\right] J_n(\lambda r)\, r dr$$

Here and in the sequel, quantities with the index (n) are coefficients in the expansion of the corresponding functions in Fourier series in terms of the angular coordinate ϕ.

The condition (1.8) allows us to express the quantity $B_n(\lambda)$ in terms of the remaining unknown functions, after which we obtain the following pair of integral equations from (1.9) and (1.10) for the basic unknowns

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* It is assumed that all of the functions can be represented by the corresponding series and integrals.

$A_n(\lambda)$*.

$$\int_0^\infty A_n(\lambda) J_n(\lambda r)\, d\lambda = \chi_n(r) \qquad (r < a),$$
$$\int_0^\infty \frac{\lambda A_n(\lambda)}{1 - g(\lambda)} J_n(\lambda r)\, d\lambda = \Psi_n(r) \qquad (r > a), \qquad g(\lambda) = \frac{\lambda h + \sinh \lambda h e^{-\lambda h}}{\lambda h + \sinh \lambda h \cosh \lambda h} \qquad (1.13)$$

Here

$$2(1-\nu)\chi_n(r) = 2\mu\, w_0^{(n)} + \frac{(x\tau_{x0} + y\tau_{y0})^{(n)}}{2(1-\nu)} - \qquad (1.14)$$
$$- \int_0^\infty [2(1-\nu) C_n \coth \lambda h + E_n] J_n(\lambda r)\, d\lambda$$

$$\Psi_n(r) = \sigma_0^{(n)} - 2\nu \left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right)_{z=0}^{(n)} + \left(x \frac{\partial^2 \Phi_1}{\partial z^2} + y \frac{\partial^2 \Phi_2}{\partial z^2}\right)_{z=0}^{(n)} +$$
$$+ \int_0^\infty \lambda \left[C_n + E_n \coth \lambda h - \frac{(1-2\nu) C_n - D_n}{\sinh^2 \lambda h}\right] J_n(\lambda r)\, d\lambda$$

$$E_n = \frac{\lambda}{2(1-\nu)} \int_0^\infty (x\tau_{x0} + y\tau_{y0})^{(n)} J_n(\lambda r)\, r dr$$

We note that, by expressing $\Psi_n(r)$ as a Hankel integral (it is to be taken as identically zero for $r < a$), the system (1.13) can be brought to the form

$$\int_0^\infty \Phi_n(\lambda) J_n(\pi r)\, d\lambda = \omega_n(r) \quad (r < a), \qquad \int_0^\infty \frac{\lambda \Phi_n(\lambda)}{1 - g(\lambda)} J_n(\lambda r)\, d\lambda = 0 \quad (r > a) \qquad (1.15)$$

Here $\omega_n(r)$ is a known function, while $\Phi_n(\lambda)$ is a new unknown quantity which is related to $A_n(\lambda)$ by the simple expression

$$\Phi_n(\lambda) = A_n(\lambda) - [1 - g(\lambda)] \int_a^\infty \Psi_n(r) J_n(\lambda r)\, r dr \qquad (1.16)$$

In the solution of Equations (1.15), we shall start from a system of a pair of integral equations of simpler form

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* Particular cases of such equations are examined in [4].

$$\int_0^\infty f_n(\lambda)\, J_n(\lambda r)\, d\lambda = F_n(r) \quad (r < a), \qquad \int_0^\infty \lambda f_n(\lambda)\, J_n(\lambda r)\, d\lambda = 0 \quad (r > a) \tag{1.17}$$

The exact solution of these equations is given by the formula [5]:

$$f_n(\lambda) = \sqrt{\frac{\pi\lambda}{2}} \int_0^a t^{1/2} \varphi_n(t)\, J_{n-1/2}(\lambda t)\, dt \tag{1.18}$$

where

$$\frac{\pi}{2} t^n \varphi_n(t) = \lim_{x\to 0} [x^n F_n(x)] + t \int_0^t (t^2 - x^2)^{-1/2} [x^n F_n(x)]'\, dx \tag{1.19}$$

We introduce the notation

$$\frac{\Phi_n(\lambda)}{1 - g(\lambda)} = f_n(\lambda)$$

Then the system (1.15) can be brought into the form (1.17), whereby the right side will contain the unknown function $\Phi_n(\lambda)$:

$$F_n(r) = \omega_n(r) + \int_0^\infty g(\lambda)\, \Phi_n(\lambda)\, J_n(\lambda r)\, d\lambda \tag{1.20}$$

The substitution of (1.20) into (1.19) gives

$$\frac{\pi}{2} t^n \varphi_n(t) = \lim_{x\to 0} [x^n \omega_n(x)] + t \int_0^t x^{n-1} (t^2 - x^2)^{-1/2} [n\omega_n(x) + x\omega_n'(x)]\, dx +$$

$$+ t \int_0^t x^n (t^2 - x^2)^{-1/2} dx \int_0^\infty g(\lambda)\, f_n(\lambda) J_{n-1}(\lambda x)\, d\lambda \tag{1.21}$$

With (1.18) taken into account this relation is an integral equation for the function $\phi_n(t)$. After the change of variables $x = t \sin\theta$ and the use of the Sonine integral [6]

$$J_{n-1/2}(z) = \sqrt{\frac{2z}{\pi}} \int_0^{\pi/2} J_{n-1}(z \sin\theta) \sin^n\theta\, d\theta \tag{1.22}$$

Equation (1.21) can be brought into the form

$$\varphi_n(t) = \frac{2}{\pi} \left[F_n(t) + \int_0^1 M_n(x, t)\, \varphi_n(x)\, dx \right] \tag{1.23}$$

The kernel and the nonhomogeneous term of this equation have the forms

$$M_n(x, t) = \pi \sqrt{xt} \int_0^\infty g(\lambda) J_{n-1/2}(\lambda x) J_{n-1/2}(\lambda t) \lambda d\lambda$$

$$t^n F_n(t) = \lim_{x\to 0} [x^n \omega_n(x)] + nt^n \int_0^{1/2\pi} \omega_n(t\sin\theta) \sin^{n-1}\theta d\theta +$$

$$+ t^{n+1} \int_0^{1/2\pi} \omega_n'(t\sin\theta) \sin^n \theta d\theta \qquad (1.24)$$

Hence, the problem which has been posed reduces to a Fredholm integral Equation (1.23) with a symmetric kernel, the solution of which yields the function $f_n(\lambda)$ to be determined by means of **Formula** (1.18).

2. Contact problem for the elastic layer in the absence of friction. We apply the results obtained to the solution of the following problem: a plane circular stamp, which is rigid in its plane, is impressed on an elastic layer lying on a rigid base, but with a line of action of the force that does not coincide with the axis of the stamp (figure). If friction is neglected, both between the stamp and the layer and between the layer and the base, then the shear stresses on the boundaries of the layer are zero. In addition, normal displacements on the plane $z = h$ and normal stresses in the region $z = 0$, $r > a$ are absent.

Hence, this problem is a particular case of the problem examined in Section 1 for

$$\tau_{xh} = \tau_{yh} = w_h = \tau_{x0} = \tau_{y0} = \sigma_0 = 0, \qquad w_0 = \delta + \gamma x \qquad (2.1)$$

where δ is the translational displacement of the stamp along the $0z$-axis and γ is the angle of rotation about the $0y$-axis.

It is not difficult to show that in the problem under consideration $\Phi_1 = \Phi_2 = 0$, and in the expressions (1.1) it is necessary to set $C_n = D_n = 0$, $B_n = (1 - 2\nu - \lambda h \coth \lambda h) A_n$ for the functions Φ_3 and Φ_4. Further, from the form of the function w_0 it follows that it is necessary to retain only the terms with $n = 0$ and $n = 1$ in the Fourier series expansions. In connection with this, the problem may be formally broken up into two problems: an axially-symmetric problem in which all of the unknown functions are proportional to the quantity δ, and a problem associated with the rotation of the stamp wherein all of the quantities contain the factor $\gamma \cos \phi$. Since the first problem has already been investigated in [1], we turn to the second problem, in which the pair of equations (1.13) for $A_1 \equiv A$ have the form

$$\int_0^\infty A(\lambda) J_1(\lambda r)\, d\lambda = \frac{\mu\gamma}{1-\nu} r \quad (r < a), \qquad \int_0^\infty \frac{\lambda A(\lambda)}{1-g(\lambda)} J_1(\lambda r)\, d\lambda = 0 \quad (r > a) \tag{2.2}$$

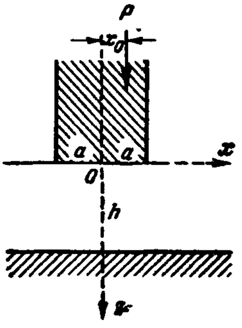
In accordance with (1.23) - (1.24), this system may be reduced to a Fredholm integral equation. Assuming in (1.18), (1.23), and (1.24) that

$$n = 1, \qquad \omega_1(x) = \frac{\gamma\mu}{1-\nu} x$$

we obtain for the unknown quantity $A(\lambda)$

$$A(\lambda) = [1 - g(\lambda)] \int_0^a \varphi(t) \sin \lambda t dt \tag{2.3}$$

where the function $\phi(x)$ must be found from the equation



$$\varphi(x) = \frac{4\mu\gamma x}{\pi(1-\nu)} + \frac{1}{\pi} \int_0^a [G(t-x) - G(t+x)]\, \varphi(t)\, dt$$

where (2.4)

$$G(u) = \int_0^\infty g(\lambda) \cos \lambda u\, d\lambda \tag{2.5}$$

We note that a number of quantities, and in particular the stresses along the base of the stamp, can be directly expressed in terms of the function $\phi(t)$, namely;

$$\sigma_z \Big|_{\substack{z=0 \\ r<a}} = \left[r \int_r^a \frac{\varphi'(t)\, dt}{(t + \sqrt{t^2 - r^2}) \sqrt{t^2 - r^2}} - \frac{\varphi(r)}{r} - \frac{r\varphi(a)}{(a + \sqrt{a^2 - r^2}) \sqrt{a^2 - r^2}} \right] \cos \varphi \tag{2.6}$$

Hence, by comparing the moment of these forces with the quantity Px_0 we obtain a relationship between the moment Px_0 and the angle of rotation

$$Px_0 = 2\pi \int_0^a t\varphi(t)\, dt \tag{2.7}$$

For the numerical calculations the basic integral equation is brought to a dimensionless form

$$\omega(\xi) = \xi + \frac{1}{\pi} \int_0^1 \omega(\tau) [K(\tau - \xi) - K(\tau + \xi)]\, d\tau, \qquad (0 \leqslant \xi \leqslant 1) \tag{2.8}$$

by means of the substitutions

$$\xi = \frac{x}{a}, \qquad \tau = \frac{t}{a}, \qquad \omega(\xi) = \frac{\pi(1-\nu)}{4\mu\gamma}\varphi(x), \tag{2.9}$$

where

$$K(u) = p\int_0^\infty \frac{\alpha + e^{-\alpha}\sinh\alpha}{\alpha + \sinh\alpha\cosh\alpha}\cos\alpha p u\, d\alpha \left(p = \frac{a}{h}\right) \tag{2.10}$$

In [1] the function $K(u)$ was tabulated for a wide range of the values of the basic parameter – ratio of the radius of the stamp to the layer thickness (see Table 1, which supplements the values in [1]), after which Equation (2.8) was solved by means of a reduction to a system of algebraic equations. The results of the corresponding calculation when

TABLE 1. **Values of the kernel $K(u)$**

u	$p = 2.5$	$p = 3.0$	u	$p = 2.5$	$p = 3.0$
0.0	2.9186	3.5022	1.1	0.1690	0.1320
0.1	2.7988	3.2973	1.2	0.1360	0.1106
0.2	2.4741	2.7670	1.3	0.1137	0.0948
0.3	2.0326	2.1032	1.4	0.0975	0.0825
0.4	1.5707	1,4813	1.5	0.0850	0.0726
0.5	1.1578	0.9922	1.6	0.0752	0.0642
0.6	0.8269	0.6504	1.7	0.0672	0.0570
0.7	0.5816	0.4296	1.8	0.0605	0.0511
0.8	0.4098	0.2944	1.9	0.0545	0.0459
0.9	0.2946	0.2130	2.0	0.0495	0.0414
1.0	0.2186	0.1632			

the interval (0.1) is divided into 10 parts is given in Table 2 ($\omega_\gamma \equiv \omega$).* In Table 3 are given the values of the coefficient

$$\psi = \frac{1-\nu}{8\mu a^3}\frac{Px_0}{\gamma}$$

which characterizes the ratio of the moment of the external force to the angle of rotation of the stamp and which is calculated according to the formula resulting from (2.7):

$$\psi = \int_0^1 \tau\omega(\tau)\, d\tau \tag{2.11}$$

* We correct here essential misprints in [3].

TABLE 2. Value of the coefficient

τ	$p = 0.5$	$p = 1.0$	$p = 1.5$	$p = 2.0$	$p = 2.5$	$p = 3.0$
			$\omega_{\gamma}(\tau)$			
0.1	0.1038	0.1249	0.1638	0.2118	0.2628	0.3130
0.2	0.2076	0.2490	0.3249	0.4185	0.5175	0.6173
0.3	0.3113	0.3716	0.4805	0.6148	0.7580	0.9073
0.4	0.4149	0.4920	0.6283	0.7961	0.9765	1.1615
0.5	0.5183	0.6096	0.7688	0.9581	1.1656	1.3833
0.6	0.6215	0.7241	0.8933	1.0974	1.3192	1.5515
0.7	0.7245	0.8351	1.0085	1.2124	1.4326	1.6641
0.8	0.8272	0.9426	1.1120	1.3034	1.5054	1.7158
0.9	0.9297	1.0466	1.2050	1.3728	1.5422	1.7128
1.0	1.0318	1.1479	1.2888	1.4257	1.5529	1.6715
			$\overset{\circ}{\omega}_{\delta}(\tau)$			
0.0	1.6623	2.7122	3.9095	5.1290	6.3498	7.5726
0.1	1.6610	2.7045	3.8933	5.1050	6.3200	7.5368
0.2	1.6573	2.6814	3.8447	5.035	6.2304	7.4281
0.3	1.6512	2.6435	3.7641	4.917	6.0783	7.2436
0.4	1.6428	2.5916	3.6521	4.748	5.8598	6.9772
0.5	1.6321	2.5267	3.5101	4.530	5.5699	6.6205
0.6	1.6194	2.4505	3.3404	4.261	5.2045	6.1630
0.7	1.6047	2.3648	3.1473	3.946	4.7638	5.5974
0.8	1.5882	2.2715	2.9362	3.594	4.2570	4.9286
0.9	1.5702	2.1729	2.7145	3.220	3.7087	4.1876
1.0	1.5508	2.0714	2.4902	2.845	3.1571	3.4387
			$\omega^{\circ}(\tau)$			
0.1	0.1060	0.1371	0.1897	0.2500	0.3114	0.3726
0.2	0.2118	0.2728	0.3753	0.4931	0.6135	0.7342
0.3	0.3175	0.4058	0.5527	0.7200	0.8979	1.0735
0.4	0.4230	0.5350	0.7184	0.9311	1.1538	1.3784
0.5	0.5281	0.6595	0.8691	1.1131	1.3711	1.6340
0.6	0.6328	0.7786	1.0026	1.2623	1.5399	1.8255
0.7	0.7372	0.8919	1.1182	1.3761	1.6520	1.9384
0.8	0.8410	0.9994	1.2158	1.4539	1.7049	1.9641
0.9	0.9438	1.0996	1.2978	1.5007	1.7049	1.9095
1.0	1.0468	1.1983	1.3668	1.5240	1.6687	1.8019

We note at this point that the complete solution of the contact problem that has been posed is obtained from the sum of the solution derived above and the results of the corresponding axially-symmetric problem (see [1]); however, this solution is in fact realized only when the pressure on the base of the stamp is non-negative. If the formula for the sum of the pressures caused by the rotation of the stamp as well as by its translational displacements is written down, then it turns out that the pressure becomes zero along a certain curve which is symmetric with respect to the coordinate ϕ. With the requirement that in the limiting

TABLE 3. **Values of the kernel $L(u)$**

p	0	0.5	1.0	1.5	2.0	2.5	3.0
$\varkappa$	1	1.51	2.20	2.95	3.72	4.49	5.26
$\varkappa^\circ$	1	1.62	2.48	3.40	4.34	5.28	6.20
ψ	1/3	0.347	0.397	0.472	0.559	0.650	0.743
ψ^*	1/3	0.352	0.422	0.519	0.626	0.738	0.852
x_0^*/a	1/3	0.325	0.301	0.285	0.275	0.270	0.266
x_0^*/a°	1/3	0.322	0.294	0.278	0.270	0.265	0.262

case this curve should be tangent to the contact region at the point $r = a$, $\phi = \pi$, we arrive at the following value for the lever arm (x_0^*) of the applied force P:

$$x_0^* = a\frac{\psi}{\varkappa}\frac{\omega_\delta(1)}{\omega_\gamma(1)} \tag{2.12}$$

with higher values indicating that the condition of non-negative pressure is violated.

In this formula $\omega_\delta(r)$ denotes the basic function $\omega(r)$ for the axially-symmetric case, and the coefficient κ is given by the formula [1]

$$\varkappa = \frac{1-\nu}{4\mu a}\frac{P}{\delta} = \int_0^1 \omega_\delta(\tau)\,d\tau \tag{2.13}$$

Table 3 gives values of the quantity κ and the limiting lever arm x_0^*.

In concluding this section we note that the case $p = 0$ corresponds to the well-known contact problem for the half-space. For this case $K \equiv 0$, $\omega(\xi) \equiv \xi$, $\psi = 1/3$, $x_0^* = a/3$.

3. Contact problem for a layer attached to the base. The methods developed in Section 1 of this paper also allow one to investigate other mixed problems for an elastic layer when the displacements u and v are given on the surface $z = h$ instead of the shear stresses τ_{xz} and τ_{yz}. The reduction to a pair of integral equations of the type (1.15) can also be accomplished in this case if one takes as the second additional condition (1.6) the relationship

$$(\Phi_0 + x\Phi_1 + y\Phi_2 + z\Phi_3)_{z=h} = 0 \tag{3.1}$$

The form of the system (1.15) is completely retained in this case; however the function $g(\lambda)$ turns out to depend on Poisson's ratio:

$$g(\lambda) = \frac{\alpha(\alpha+1) + 4(1-\nu)^2 - (3-4\nu)\sinh\alpha \cdot e^{-\alpha}}{\alpha^2 + 4(1-\nu)^2 + (3-4\nu)\sinh^2\alpha}, \qquad \alpha = \lambda h \tag{3.2}$$

Thus, in the case of the contact problem for the layer attached to the base (friction between the stamp and the layer is neglected as before), calculations can be carried out according to the same scheme as in Section 2.

TABLE 4. **Values of the kernel $L(u)$**

u	$p = 0.5$	1.0	1.5	2.0	2.5	3.0
0.0	0.6882	1.3765	2.0648	2.7530	3.4412	4.1295
0.1	0.6867	1.3641	2.0232	2.6554	3.2528	3.8085
0.2	0.6820	1.3277	1.9042	2.3864	2.7578	3.0123
0.3	0.6744	1.2695	1.7244	2.0082	2.1222	2.0934
0.4	0.6638	1.1932	1.5062	1.5950	1.5112	1.3287
0.5	0.6506	1.1031	1.2734	1.2090	1.0205	0.8052
0.6	0.6348	1.0041	1.0467	0.8858	0.6710	0.4890
0.7	0.6167	0.9008	0.8412	0.6358	0.4418	0.3132
0.8	0.5966	0.7975	0.6644	0.4532	0.3000	0.2188
0.9	0.5748	0.6978	0.5187	0.3260	0.2158	0.1646
1.0	0.5516	0.6045	0.4026	0.2400	0.1647	0.1310
1.1	0.5272	0.5192	0.3126	0.1836	0.1315	0.1091
1.2	0.5020	0.4429	0.2445	0.1459	0.1092	0.09246
1.3	0.4764	0.3759	0.1938	0.1198	0.09352	0.07848
1.4	0.4504	0.3179	0.1566	0.1011	0.0814	0.06804
1.5	0.4244	0.2684	0.1295	0.08732	0.07090	0.05979
1.6	0.3988	0.2266	0.1094	0.07704	0.06220	0.05214
1.7	0.3735	0.1918	0.09414	0.0688	0.0555	0.04581
1.8	0.3489	0.1630	0.08228	0.06164	0.04982	0.04137
1.9	0.3252	0.1393	0.07292	0.005518	0.04450	0.03720
2.0	0.3022	0.1200	0.06549	0.04976	0.03982	0.03315

In Table 4 values of the basic kernel are given

$$L(u) = p\int_0^\infty \frac{\alpha(\alpha+1) + 4(1-\nu)^2 - (3-4\nu)\sinh\alpha e^{-\alpha}}{\alpha^2 + 4(1-\nu)^2 + (3-4\nu)\sinh^2\alpha} \cos\alpha p u d\alpha \tag{3.3}$$

for various values of $p = a/h$. $\nu = 0.3$ is assumed.

For the same values of the parameters, the quantities $\omega_\delta^\circ(r)$, $\omega_\gamma^\circ(r)$, κ°, ψ°, $x_0^{*\circ}$, are given in Tables 2 and 3; here the index indicates that the data refer to the case when the layer and the base are attached to one another.

The numerical results that have been obtained in this paper allow one, in particular, to assess the influence of the thickness of the layer on the indentation of the stamp in two limiting cases – when friction between the layer and the base is absent and when the layer and the base are attached together.

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